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On a class of quartic surfaces and an associated integrable differential system

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Abstract
Given a unicursal quartic in a plane and four bi-tangential straight lines, there exist a pair of conics four times tangential to it. Choosing an arbitrary point K_0 outside the plane, the lines and either one of the conics determine four planes and a cone, and a quartic surface can be found that stays tangential to them and presents 15 conic point singularities, including K_0. We introduce a one-parameter family of symmetric tensors which are polynomial functions of degree 7 of the Cartesian coordinates. Each of them determines two families of curves on the surface, which are the null curves of the associated metric and are the integral curves of a differential system which turns out to possess the Painlevé property and to be integrable by quadratures. Differential systems of that form have been found (Gaffet 2006 *J. Phys. A: Math. Gen.* **39** 99) to represent the rotating motion with precession of gas clouds expanding into a vacuum.

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1. Introduction

It has recently been shown (Gaffet 2001) that the adiabatic expansion into a vacuum of a rotating mass of gas, governed by the well-known Euler equations of gasdynamics, gives rise under certain conditions—such as the monatomicity of the gas—to a Hamiltonian system of the Liouville type (Whittaker 1959), and must accordingly be integrable by quadratures. This system has also been shown to have the Painlevé property in various sub-cases.

Gaffet (2003), under the restricting assumption of the vanishing of one of the constants of the motion, has shown that the Liouville tori corresponding to a given set of integrals of the motion can be changed through an appropriate transformation of variables into a quartic surface (Σ) presenting a number of conic point singularities. In those cases, the system possesses the Painlevé property, and a solution by separation of variables was also found.

More recently, forsaking the above-mentioned constraint on the constants of motion, the system has been found (Gaffet 2006) to involve a new essential geometrical ingredient: a quartic curve, denoted as (A_4), traced on (Σ). It is one of the purposes of the present work to clarify the geometrical relation between that curve and the surface (Σ); they are in fact intimately related: knowledge of one determines the other, as it will turn out.

Given these two key geometrical elements, several polynomial functions can be defined on the surface, which constitute the coefficients of a differential system, which is integrable by quadratures. The system is shown here to have the Painlevé property when the independent variable is chosen to coincide with the thermasy of the cloud, i.e. the time integral of its absolute temperature: $u = \int T\,\mathrm{d}t$. The Painlevé property (Painlevé 1902, Ince 1956) has long been known to be associated with complete integrability, after the works of Kowalevski (1889a, 1889b), and Ablowitz and Segur (1977, 1980).

2. Quartic surfaces with 15 conic singular points

2.1. General properties

The quartic surfaces considered in the present work present 15 conic singularities and, after a linear coordinate transformation that sends one of the conic points to infinity, their equation assumes the form

$$A_2\rho^2 + B_3\rho + C_4 = 0 \tag{2.1}$$

where A_2, B_3, C_4 are polynomials in the coordinates (ξ, η), ρ is the third coordinate and the lower index is a reminder of the polynomial's degree. The second-degree curve $A_2(\xi, \eta) = 0$, denoted as (A_2), is the trace on horizontal sections (ρ = constant) of the vertical cylinder tangential to (Σ) at the conic point at infinity. We further require that the discriminant

$$\Delta(\xi, \eta) = B_3^2 - 4A_2C_4 \tag{2.2}$$

be decomposable into a product of four linear factors and one quadratic factor

$$\Delta \equiv \Pi_1\Pi_2\Pi_3\Pi_4E \tag{2.3}$$

which represent respectively vertical planes denoted as (Π_a) $(a = 1, \ldots, 4)$ and a vertical cylinder (E), tangential to the surface all along their intersection with it.

The linear factors determine a quadrangle in the (ξ, η) plane, with three diagonals; there is no loss of generality in choosing the coordinate system in such a way that one of the diagonals is removed to infinity, and then, e.g., (Π_1) and (Π_2) are parallel, and (Π_3) and (Π_4) are parallel too; the intersections K_{12} of (Π_1) and (Π_2), and K_{34} of (Π_3) and (Π_4), are conic singular points at infinity. Finally, we also require that two horizontal plane sections (i.e. sections by planes passing through K_{12} and K_{34}) share with $(\Pi_1 \cdots \Pi_4)$ the property of being tangential to the surface all along their intersection with it, e.g., and without loss of generality, the sections $\rho = 0$ and $\rho = 1$. As a result, the following must be perfect squares:

$$C_4 \equiv \gamma_0^2 \qquad A_2 + B_3 + C_4 \equiv \gamma_5^2 \tag{2.4}$$

where γ_0 and γ_5 are second-degree polynomials; and the equation of the surface then reads

$$A_2\rho^2 + \rho\left(\gamma_5^2 - \gamma_0^2 - A_2\right) + \gamma_0^2 = 0. \tag{2.5}$$

It follows that, whenever $\Delta = 0$, $A_2 = (\gamma_0 \pm \gamma_5)^2$. Letting

$$\gamma_A = \gamma_0 + \gamma_5 \qquad \gamma_B = \gamma_0 - \gamma_5 \tag{2.6}$$

we have

$$\Delta \equiv \left(\gamma_A^2 - A_2\right)\left(\gamma_B^2 - A_2\right) \tag{2.7}$$

and, in fact, we shall assume that the following identifications hold:

$$A_2 \equiv \gamma_A^2 - \Pi_1 \cdots \Pi_4 \tag{2.8a}$$

$$\equiv \gamma_B^2 - E \tag{2.8b}$$

γ_B must then be a linear factor. The above identities mean that (A_2) is tangential to $(\Pi_1), \ldots, (\Pi_4)$ and bi-tangential to (E)—in agreement with the fact that any curve traced on (Σ) should be tangential to the locus $\Delta = 0$, which is the 'boundary' of (Σ) in projection on the (ξ, η) plane.

Let us finally remark that the conic points lie at the 14 intersections of $(\Pi_1), \ldots, (\Pi_4)$ and (E); and the 15th is the projection point, at infinity along the vertical (ρ) axis. One may verify that the decomposability of the discriminant in the form (2.3) is preserved, independently of the choice of conic point as the projection point.

2.2. Reduced form of the surface

Through a linear transformation of the coordinates (ξ, η), it is possible to change a given quadrangle into any other quadrangle. It will be convenient to consider the new coordinate system (u, v) in which the linear factors Π_i have the following reduced form:

$$\begin{aligned} \Pi_1 &\equiv u - 1 - v \\ \Pi_2 &\equiv u + 1 - v \\ \Pi_3 &\equiv u - 1 + v \\ \Pi_4 &\equiv u + 1 + v \end{aligned} \tag{2.9}$$

so the three diagonals of the quadrangle are the coordinate axes and the line at infinity.

There exists a one-parameter family of conics (A_2) tangential to that quadrangle, and they are given by equation (2.8a), in which

$$\gamma_A \equiv u^2 - v^2 - \lambda \tag{2.10}$$

and λ is the free parameter. Similarly, the conic (E) is given by (2.8b), where γ_B involves three *a priori* arbitrary parameters b_1, b_2, b_3:

$$\gamma_B \equiv su + \mathrm{d}v + b_3 \tag{2.11}$$

with

$$s \equiv b_1 + b_2 \qquad d \equiv b_1 - b_2. \tag{2.12}$$

Thus, the reduced form of the surface involves four parameters in all: $b_{1,2,3}$ and λ.

2.3. The unicursal quartics (A_4) *associated with* (Σ)

When the four parameters satisfy a certain constraint

$$F(\lambda, b_1, b_2, b_3) = 0 \tag{2.13}$$

there exists a second-degree surface (Σ_2)—a quadric—whose intersection with (Σ) decomposes into a pair of quartic curves, (A_4) and (A_4') say. As it turns out, they are *unicursal*, and their horizontal projections consequently present three double points. Being curves traced on the surface (Σ), the projections are, in addition, bi-tangential to $(\Pi_1), \ldots, (\Pi_4)$ and four times tangential to (E) (see section 2.1)

Conversely, let (A_4) be a plane unicursal quartic, bi-tangential to each side of the quadrangle: these curves constitute a family depending on three angular parameters $\alpha_1, \alpha_2, \alpha_3$. This shows that the relation $F(\lambda, b_1, b_2, b_3) = 0$ can be parametrized by the three α.

We first give the parametric representation of the family of bi-tangential quartics in terms of $\alpha_{1,2,3}$ and then the expressions of λ, b_1, b_2, b_3 in terms of the α, i.e., the parametric representation of (A_4) and (E).

To start with, A_4 must have the form, analogous to (2.8a),

$$A_4 \equiv Q_2^2 - 4\Pi_1\Pi_2\Pi_3\Pi_4 \tag{2.14}$$

where Q_2 is of second degree:

$$Q_2 \equiv (q_{11}u^2 + q_{22}v^2 + q_{33}) + 2(q_{12}uv + q_{13}u + q_{23}v). \tag{2.15}$$

Writing for short ch_i, sh_i for $\cosh\alpha_i$, $\sinh\alpha_i$ ($i = 1, 2, 3$), we introduce new hyperbolic angles β_i as

$$\left.\begin{aligned}\cosh\beta_i &= (1 + ch_j ch_k)/(ch_j + ch_k)\\ \sinh\beta_i &= -sh_j sh_k/(ch_j + ch_k)\end{aligned}\right\} \quad (i, j, k = \text{Permutation of } 1, 2, 3) \tag{2.16}$$

together with the sums of angles

$$\varphi_i = \alpha_i + \beta_i. \tag{2.17}$$

Then, the coefficients of Q_2 are the following:

$$\left.\begin{aligned}q_{ii} &= 2ch_i - \Omega\\ q_{ij} &= (ch_i + ch_j)\sinh\varphi_k\end{aligned}\right\} \quad (i, j, k = \text{Permutation of } 1, 2, 3) \tag{2.18}$$

where Ω is the symmetrical function:

$$\Omega = (ch_1 + ch_2 + ch_3) + ch_1ch_2ch_3 - sh_1sh_2sh_3. \tag{2.19}$$

To find the coefficients λ, b_1, b_2, b_3 of the corresponding surface (Σ), it turns out to be convenient to introduce a fourth angular variable α_0, defined by

$$\cosh^2\alpha_0 \equiv -(ch_1ch_2 + ch_2ch_3 + ch_3ch_1) \tag{2.20}$$

and, writing from now on ch_0, sh_0 for $\cosh\alpha_0$, $\sinh\alpha_0$, we find

$$\begin{aligned}\lambda &= (R+1)/(R-1)\\ R &= (ch_0 - ch_2)/(ch_0 + ch_1)\end{aligned} \tag{2.21}$$

$$\begin{aligned}s &\equiv b_1 + b_2 = (1+\lambda)sh_1/sh_0\\ d &\equiv b_1 - b_2 = (1-\lambda)sh_2/sh_0\\ b_3 &\equiv 2\, sh_3/sh_0\end{aligned} \tag{2.22}$$

(We note that some of the ch_i may take negative values, while leaving the corresponding sh_i real, as well as ch_0 and sh_0).

Conversely, given λ, b_1, b_2, one deduces explicit expressions for the ratios $x_i \equiv ch_i/ch_0$ and for ch_0^2. First, we obtain

$$x_3 \equiv ch_3/ch_0 = \frac{\left(\lambda^2 + 1 - b_1^2 - b_2^2\right)}{2(\lambda - b_1b_2)} \tag{2.23}$$

and then, recalling the definition (2.21) of λ, which we rewrite as

$$\lambda = (2x_3 + x_2 - 1)/(x_2 + 1), \tag{2.24}$$

x_2 can be deduced, as well as x_1:

$$x_1 = -(1 + x_2x_3)/(x_2 + x_3). \tag{2.25}$$

Next, we find

$$ch_0^2 = \frac{(\lambda - b_1 b_2)^2}{\left(b_1^2 - 1\right)\left(b_2^2 - 1\right)} \tag{2.26}$$

and hence (see (2.22)) an explicit expression of b_3^2:

$$b_3^2 = b_3^2(\lambda, b_1, b_2).$$

The above equation is the constraint (2.13) on the four coefficients of the reduced form of (Σ), and we assume it holds, in the remainder of the present work. Introducing auxiliary quantities

$$\begin{aligned} \hat{s} &\equiv (R-1)^2 s^2 \\ \hat{d} &\equiv (R-1)^2 d^2 \\ \hat{b} &\equiv (R-1)\left(b_3^2 - 4\right), \end{aligned} \tag{2.27}$$

the constraint also reads

$$\frac{R}{\hat{s} - 4R^2} = \frac{1}{\hat{d} - 4} + \frac{1}{\hat{b}}. \tag{2.28}$$

2.4. The symmetry group of the surface

The reduced form of the surface is preserved by several linear transformations of the (ξ, η) or (u, v) plane: for instance, the inversion $v' = -v$ of the second coordinate results in the following transformation, denoted as (T_0), of the parameters:

$$(T_0) : \lambda' = \lambda, \qquad b_1' = b_2, \qquad b_2' = b_1, \qquad b_3' = b_3. \tag{2.29}$$

Similarly, exchanging the axes u, v, i.e. exchanging these two diagonals of the quadrangle, induces the transformation

$$(T_1) : \lambda' = -\lambda, \qquad s' = d, \qquad d' = s, \qquad b_3' = b_3. \tag{2.30}$$

Another possibility is the exchange of one coordinate axis with the line at infinity—which is the third diagonal of the quadrangle:

$$(T_2) : \lambda' = \frac{3-\lambda}{\lambda+1}, \qquad s' = \frac{-2b_3}{\lambda+1}, \qquad d' = \frac{2d}{(\lambda+1)}. \tag{2.31}$$

Combining (T_1) and (T_2) yields a total of six distinct values λ_i of λ; and the six quartets $(-1, +1, \lambda_i, \infty)$ are all homographically related.

There exists in addition a transformation that does not preserve the identity of the conic point at infinity on the ρ-axis, which is

$$(T_3) : \lambda' = b_1, \qquad b_1' = \lambda, \qquad b_2' = b_2, \qquad b_3' = b_3, \tag{2.32}$$

i.e., it exchanges the roles of λ and b_1, just as (T_0) exchanges the roles of b_1, b_2.

Combining all these transformations, one can place any of the 15 conic points at infinity, and the corresponding parameters λ, b_1, b_2 of the surface may thus be deduced. The transformation group has $6 \times 15 = 90$ elements, since λ takes six different values for each choice of a conic point. Symmetrical combinations of $ch_{1,2,3}$, such as

$$X \equiv sh_0^2, \qquad Y \equiv (ch_1 + ch_2 + ch_3)/ch_0, \qquad Z \equiv ch_1 ch_2 ch_3 \big/ ch_0^3, \tag{2.33}$$

can take only 15 distinct values : one for each choice of conic point.

The group admits three invariants, which are rational functions of X, Y, Z and also rational functions of S, K, P:

$$\begin{aligned} S &\equiv \lambda^2 + b_1^2 + b_2^2 \\ K &\equiv \lambda^2 b_1^2 + \lambda^2 b_2^2 + b_1^2 b_2^2 \\ P &\equiv \lambda b_1 b_2. \end{aligned} \tag{2.34}$$

The first two are I_0^2 and I_1^2, where

$$I_0 \equiv \frac{ch_0^3}{sh_0}(Y + Z) \tag{2.35}$$

$$I_1^2 \equiv (4X + 1)^2/4 - \frac{(X + 1)}{X} Y[(3X + 1)Y + (X + 1)(4X + 1)Z]. \tag{2.36}$$

The third one, I_2, can conveniently be expressed in terms of

$$\begin{aligned} \sigma &\equiv \frac{(4X + 1)}{2I_1} \\ \tau &\equiv [(X + 1)Y^2 - (4X + 1)^2/4] \big/ I_1^2 \end{aligned} \tag{2.37}$$

as

$$\frac{1/2}{(\tau - \sigma I_2/I_1)} \equiv \frac{\tau}{(\tau^2 - \sigma^2)} - \frac{1}{(\tau + 1)}. \tag{2.38}$$

Given the three invariants, the 15 sets of values of ch 1, 2, 3 may be deduced. It is worth noting that changing the signs of Y and Z, without changing $ch_{1,2,3}$, transforms (A_2) and (E) into new conics $(\tilde{A}_2)$, $(\tilde{E})$, without altering (A_4). Consequently, (A_4) is four times tangential to both (E) and $(\tilde{E})$. Remarkably, this is also true of (A_4').

3. The symmetric tensors *X* and *G*

Given a quartic surface (Σ) of the type described in the preceding section, it is possible to construct two symmetric tensors X_{ij}, G_{ij}, defined on the plane (ξ, η). The first one, X, is uniquely determined, but the other one, G, belongs to a family depending quadratically on a parameter, z. The symmetric tensor G determines null curves in the (ξ, η) plane, which are the solutions of the differential system

$$G_{ij}\, \mathrm{d}\xi^i\, \mathrm{d}\xi^j = 0 \tag{3.1}$$

(where ξ^1, ξ^2 stands for ξ, η). It is this differential system that we will show to be integrable by quadratures, and to possess the Painlevé property, with respect to a certain independent variable u (We do not make use in what follows of the reduced coordinates u, v). As mentioned in the introduction, it is a differential system of the same type that describes the evolution of a certain class of gas clouds, expanding with rotation and precession (Gaffet 2006), and the independent variable u then has the physical meaning of the thermasy of the cloud, which is the time integral of its absolute temperature: $u = \int T\,\mathrm{d}t$.

3.1. The symmetric tensor X

The tensors X and G have in common the essential property that their components are *polynomial functions* of the coordinates (ξ, η). X is characterized by the following properties:

(a) X_{ij} is a fourth-degree polynomial function of (ξ, η).
(b) Along any straight line of direction p^i, the component $X_{ij}p^ip^j$ is only quadratic; in particular, the dependence of X_{11} on ξ^1 is quadratic, as well as that of X_{22} on ξ^2.
(c) Whenever the discriminant Δ vanishes

$$X^{ij}(\partial_i\Delta)(\partial_j\Delta) = 0 \tag{3.2}$$

where the symbol ∂_k indicates partial differentiation $\partial/\partial\xi^k$ (Indices can be raised and lowered by means of the Levi-Civita symbol, see, e.g., Chandrasekhar (1983)).
(d) Whenever $\Delta = 0$, the norm

$$X^{ij}X_{ij} = -2A_6 \tag{3.3}$$

where A_6 denotes the product

$$A_6 \equiv A_2A_4. \tag{3.4}$$

As we now show, these four conditions fully determine X. First, we remark that, at conic points (which are the double points of the locus $\Delta = 0$), equation (3.2) gives two independent conditions, while equation (3.3) gives a third, so that all three components X_{ij} can be determined.

Thus, at a conic point K_{ab}, where the slopes of the two tangents are $\mathrm{d}\eta/\mathrm{d}\xi = m_a$ and m_b, one finds

$$\begin{aligned} X_{11} &= 2\sqrt{A_6}m_am_b/(m_a - m_b) \\ X_{12} &= -\sqrt{A_6}(m_a + m_b)/(m_a - m_b) \\ X_{22} &= 2\sqrt{A_6}/(m_a - m_b). \end{aligned} \tag{3.5}$$

Further, from equations (2.8a) and (2.14), one can substitute a rational expression for the radical

$$\sqrt{A_6} = \gamma_A Q_2. \tag{3.6}$$

Owing to the restrictions (a) and (b) on the degree of X_{ij}, knowledge of its value at the 14 conic points turns out to be sufficient for its determination all over the plane, by means of the Lagrange polynomial interpolation method. In particular, one can calculate the norm, which is found to be generally given by

$$X \cdot X = -2\left(A_6 + k_0^2\Delta\right) \tag{3.7}$$

where k_0 is a constant, which turns out to be

$$k_0 = sh_0 \tag{3.8}$$

(In what follows, we denote by $A \cdot B$ the scalar contraction $A^{ij}B_{ij}$ of a pair of symmetric tensors).

3.2. *The symmetric tensor G*

Another symmetric tensor G can be defined in a way analogous to X by the following conditions:

(a) G_{ij} is a polynomial function of (ξ, η) of degree 7.
(b) Along any straight line of direction p^i, the component $G_{ij}p^ip^j$ is only of the fifth degree.
(c) Whenever $\Delta = 0$,

$$G^{ij}(\partial_i\Delta)(\partial_j\Delta) = 0. \tag{3.9}$$

(d) The norm

$$G \cdot G = -G_3^2\Delta\big/2 \tag{3.10}$$

where the scalar G_3 is a cubic polynomial.

(e) The scalar product

$$G \cdot X = -k_0 G_3 \Delta. \tag{3.11}$$

From the conditions (c) and (d), it follows that $G^{ik}\partial_k \Delta = 0$ when $\Delta = 0$. Hence, the three components G_{ij} are proportional to each other along each line (Π_a) of the quadrangle. Further, the proportionality constant, being the slope of the line, takes two distinct values at conic points, so that the three G_{ij} must vanish there. As there are five conic points on each line, the components must have the form, on (Π_a),

$$G_{ij} = (\partial_i \Pi_a)(\partial_j \Pi_a)\Pi_b \Pi_c \Pi_d \, E \, \gamma_{2a} \tag{3.12}$$

where a, b, c, d is a permutation of 1, 2, 3, 4 and γ_{2a} is a quadratic factor

$$\gamma_{2a} = c_{a2}\xi^2 + c_{a1}\xi + c_{a0}. \tag{3.13}$$

Similarly, it may be seen that, on (E), the components must read

$$G_{ij} = (\partial_i E)(\partial_j E)\Pi_a \Pi_b \Pi_c \Pi_d \, T \tag{3.14}$$

where $T(\xi, \eta)$ is a linear factor

$$T = c_{50} + c_{51}\xi + c_{52}\eta. \tag{3.15}$$

From the data of G_{ij} on the quadrangle and on (E), together with the constraints (a) and (b) on its degree, it is possible to deduce its value all over the plane, using Lagrange interpolation; and the resulting expression linearly depends on the 15 coefficients $c_{A0,1,2}$ $(A = 1, \ldots, 5)$.

Next, it can be seen that the condition (d), which involves a squared factor G_3^2, entails that γ_{2a} should in fact be a perfect square and, similarly, that T should represent a straight line *tangential* to (E). Thus, $\gamma_{2a} \equiv \gamma_{1a}^2$, with

$$\gamma_{1a} = \alpha_{a1}\xi + \alpha_{a0} \tag{3.16}$$

and, introducing a parametric representation of the conic (E), with parameter τ, the factor T involves a perfect square as well:

$$T = \gamma_{15}^2 \big/ D_2(\tau) \tag{3.17a}$$

where

$$\gamma_{15} = \alpha_{51}\tau + \alpha_{50} \tag{3.17b}$$

and D_2 is some quadratic polynomial. This new representation involves ten parameters $\alpha_{A0,1}$ only—but the dependence of G_{ij} on the parameters is quadratic.

Having thus determined the form of the components G_{ij}, the scalar factor G_3 may be deduced, using either the linear equation (3.11) or the quadratic one (3.10). From equation (3.10), it is found that G_3^2 admits the following simple expression at conic point K_{ab}, the intersection of lines (Π_a) and (Π_b),

$$G_3^2 = \varphi_{ab}\gamma_{2a}(K_{ab})\gamma_{2b}(K_{ab}) \tag{3.18}$$

where φ_{ab} is a known constant. Then, taking the square root, G_3 is found as

$$G_3 = \psi_{ab}\gamma_{1a}\gamma_{1b} \tag{3.19}$$

where $\varphi_{ab} = \psi_{ab}^2$. An expression of the same form also holds at conic points K_{an} $(n = 5, 6)$ which are the intersections of (Π_a) and (E), with the factor γ_{1b} replaced by γ_{15}. Now, on (Π_a), G_3, being cubic, must have the form

$$G_3 = \gamma_{1a} g_{2a} \tag{3.20}$$

where the factor g_{2a} is quadratic; so we find, at the five conic points on (Π_a),

$$g_{2a} = \psi_{an}\,\gamma_{1n} \qquad (n = 1, \ldots, 6, n \neq a) \tag{3.21}$$

(where γ_{1n} means γ_{15} whenever $n > 4$). As three values are sufficient to fully determine the trinomial g_{2a}, the above equation gives two *linear* constraints on the coefficients $\alpha_{A0,1}$. Applying it to the four lines of the quadrangle yields a total of six independent linear constraints, so that all the coefficients can be linearly determined in terms of four of them only.

Alternatively, G_3 can be determined from the linear equation (3.11). The resulting expression, being linear in the 15 coefficients $c_{A0,1,2}$, is quadratic in $\alpha_{A0,1}$.

Equating this with the independently obtained expression (see (3.20) and (3.21))

$$G_3 = \psi_{an}\gamma_{1a}\gamma_{1n} \tag{3.22}$$

gives a quadratic relation on the coefficients, one for each conic point. It turns out that two such equations only are independent, so that two of the coefficients remain undetermined.

As the system is homogeneous, one of the coefficients may always be chosen arbitrarily without loss of generality, so that there is effectively a one-parameter family of solutions only, with parameter z, say. For a given z, one would *a priori* expect four distinct solutions to be present, but in fact, as the full system is an overdetermined one, one solution only is found to exist. All coefficients $\alpha_{A0,1}$ turn out to be linear functions of the parameter z; and so the z-dependence of the tensor G is quadratic.

An essential element of the above method of determining the tensor G was the consideration of the constants ψ_{an} which, being square roots, have an *a priori* undetermined sign. This arbitrariness in fact does not exist, as no solution will be found to the system unless one makes either of only two acceptable choices of sign. Of these two possibilities, only one leads to a differential system integrable by quadrature in the way described below. It is the case that we consider in what follows.

3.3. The differential system and its solution by quadrature

The symmetric tensor G, which may be viewed as a bi-dimensional metric, determines null curves on the (ξ, η) plane, which are the solutions of the differential equation (3.1):

$$G_{ij}\,\mathrm{d}\xi^i\,\mathrm{d}\xi^j = 0.$$

We choose to define an independent variable u through the relation

$$X_{ij}\xi'^i(u)\xi'^j(u) = -G_3\sqrt{\Delta} \tag{3.23}$$

where a prime denotes derivation with respect to u.

Equations (3.1) and (3.23) determine the derivatives ξ'^i, through their quadratic combinations:

$$\xi'^i\xi'^j = \frac{(P^{ij} + \sqrt{\Delta}Q^{ij})}{2A_6} \tag{3.24}$$

where the symmetric tensor P^{ij}

$$P^{ij} = G^{ia}X^j_a + G^{ja}X^i_a \tag{3.25}$$

has components which are polynomials of the tenth degree, while Q^{ij}, also symmetric and of degree 7, is given by

$$G_3\,\Delta\,Q^{ij} = G^{ia}P^j_a + G^{ja}P^i_a. \tag{3.26}$$

Taking account of equations (3.10) and (3.11), Q^{ij} also reads

$$Q^{ij} = G_3X^{ij} - 2k_0G^{ij}. \tag{3.27}$$

We note the following identities:

$$X \cdot P = 0 \qquad X \cdot Q = -2G_3 A_6. \tag{3.28}$$

It may be worth noting here that (3.24) is the general form of the equations of motion for a reversible Hamiltonian with a second integral quadratic in the momenta, such as, for example, the Hénon–Heiles system (1964); in which case the discriminant Δ is a sixth-degree polynomial, fully decomposable into linear factors. The Hénon–Heiles system however, being soluble by separation of variables and also soluble in terms of elliptic functions, is more closely related to the degenerate cases considered by Gaffet (2003).

The system (3.24) is soluble in the following way by quadrature

$$(\xi'^2\, \mathrm{d}\xi^1 - \xi'^1\, \mathrm{d}\xi^2)/\sqrt{\Delta} = \mathrm{d}\Phi \tag{3.29}$$

where $\mathrm{d}\Phi$ is the exact differential of a function $\Phi(\xi^1, \xi^2)$ which, by construction, remains constant along integral curves.

The condition of existence of Φ is

$$\partial_a(\xi'^a/\sqrt{\Delta}) = 0. \tag{3.30}$$

and it is possible to rewrite it in a form where no irrational except $\sqrt{\Delta}$ occurs, by forming appropriate combinations of the vector:

$$U^i \equiv (2\xi'^i/\sqrt{\Delta})\partial_a(\xi'^a/\sqrt{\Delta}) = 0. \tag{3.31}$$

Thus, introducing

$$V^{ij} \equiv \xi'^i\xi'^j/\Delta = \frac{(P^{ij} + \sqrt{\Delta}Q^{ij})}{2\Delta A_6} \tag{3.32}$$

we find

$$V_i^k U_j \equiv -V^{ka}\lfloor \partial_i V_{ja} + \partial_j V_{ia} - \partial_a V_{ij} \rfloor = 0. \tag{3.33}$$

Separating the rational and irrational parts, one finally obtains the compatibility condition in polynomial form, such as

$$\begin{aligned} &\lfloor P^{12}\partial_1 P^{11} + P^{11}\partial_2 P^{22}\rfloor + \Delta\lfloor Q^{12}\partial_1 Q^{11} + Q^{11}\partial_2 Q^{22}\rfloor \\ &\quad = P^{11}[P^{2a}\partial_a \ln(\Delta\ A_6)] + \Delta Q^{11}[Q^{2a}\partial_a \ln(A_6)] - Q^{11}[Q^{2a}\partial_a \Delta]/2 \end{aligned} \tag{3.34}$$

in which the quantities $P^{ia}\partial_a \ln \Delta$, $P^{ia}\partial_a \ln A_6$ and $Q^{ia}\partial_a \ln A_6$ are polynomial.

These polynomial equations are found to be satisfied, so that $\mathrm{d}\Phi$ is indeed an exact differential.

3.4. Linear dependence of Φ on z

We have seen in section 3.2 that the tensor G depends quadratically on a free parameter z; we now show that the dependence of Φ on z is *linear*.

Let us recall that the differential system considered, being of second degree with discriminant Δ, determines two distinct systems of integral curves: (C^+) and (C^-), which are the solutions of (see (3.24))

$$\xi'^i_\pm \xi'^j_\pm = \frac{(P^{ij} \pm \sqrt{\Delta}Q^{ij})}{2A_6}. \tag{3.35}$$

From the rational form (equation (3.1)) of the system, which admits both ξ'_+ and ξ'_- as solutions, one easily deduces

$$k\left(\xi'^i_+\xi'^j_- + \xi'^j_+\xi'^i_-\right) = 2G^{ij} \tag{3.36}$$

(where k is a scalar) and

$$k\xi_+^{\prime i}\xi_-^{\prime j} = G^{ij} + \varepsilon^{ij} G_3\sqrt{\Delta}/2. \tag{3.37}$$

We similarly obtain

$$P \cdot P = 2A_6^2\left(\xi_+^{\prime i}\xi_{i-}^{\prime}\right)^2 \tag{3.38}$$

and hence, using (3.37),

$$P \cdot P = 2A_6^2 G_3^2\Delta\big/k^2. \tag{3.39}$$

On the other hand, from (3.25) and (3.7), (3.10) and (3.11)

$$P \cdot P = 2\lfloor(G\cdot G)(X\cdot X) - (G\cdot X)^2\rfloor = 2A_6G_3^2\Delta \tag{3.40}$$

so that $k^2 = A_6$.

Let us now show the following identity (from now on we usually omit the index + on $\xi_+^{\prime}$):

$$G^{ij} + \varepsilon^{ij}\ G_3\sqrt{\Delta}/2 = \xi^{\prime i}\left(X_a^j\xi^{\prime a} + k_0\sqrt{\Delta}\xi^{\prime j}\right). \tag{3.41}$$

The antisymmetric part of the identity is an immediate consequence of (3.23). The symmetric part reads, substituting for $\xi^{\prime i}\xi^{\prime j}$ its expression (3.24),

$$4A_6G^{ij} = \left(X_a^iP^{aj} + X_a^jP^{ai}\right) + \sqrt{\Delta}\left(X_a^iQ^{aj} + X_a^jQ^{ai}\right) + 2k_0\sqrt{\Delta}(P^{ij} + \sqrt{\Delta}Q^{ij}). \tag{3.42}$$

Using the definitions (3.25) and (3.26) of P and Q and noting that

$$G\cdot P = G\cdot Q = P\cdot Q = X\cdot P = 0 \tag{3.43}$$

we have

$$\begin{aligned}\left(X_a^iP^{aj} + X_a^jP^{ai}\right)\big/2 &= (G\cdot X)X^{ij} - (X\cdot X)G^{ij}\\ \left(X_a^iQ^{aj} + X_a^jQ^{ai}\right)\big/2 &= (G_3\Delta)^{-1}[(G\cdot X)P^{ij} - (X\cdot P)G^{ij}]\\ &= -k_0P^{ij}.\end{aligned} \tag{3.44}$$

Thus the P^{ij} terms cancel out on the right-hand side of (3.42), and it can be seen that the X^{ij} terms cancel each other as well and that the identity (3.42) is verified. Then, comparing (3.37), (3.41), the (C^-) solution is found to be linearly related to the (C^+) one as

$$A_6^{1/2}\xi_-^{\prime i} = X_a^i\xi^{\prime a} + k_0\sqrt{\Delta}\xi^{\prime i}. \tag{3.45}$$

That result has an interesting consequence on the form of the dependence of $\xi^{\prime}$ on the free parameter z. We have seen (section 3.2) that G, and hence P, Q, depends quadratically on z whereas X is independent of z. Therefore, from (3.24), $\xi^{\prime i}$ must be proportional to the square root of a second-degree polynomial in z:

$$\xi^{\prime 1} = \sqrt{p_2(z)} \qquad \xi^{\prime 2} = \sqrt{q_2(z)}.$$

Then, by application of (3.45)

$$\xi_-^{\prime 1} = c_0\sqrt{p_2(z)} + c_1\sqrt{q_2(z)}$$

where c_0, c_1 are the constants (i.e. independent of z); its square $(\xi_-^{\prime 1})^2$ must however be rational, as is $(\xi^{\prime 1})^2$—therefore either p_2 and q_2 are proportional or they are perfect squares. In the first case, the integral curves will not depend on z, which is not compatible with the fact that the differential equation (3.1) does depend on it; therefore, we conclude that $\xi^{\prime i}$ depends on z *linearly*.

This in turn entails that the 'integration constant' Φ, defined by (3.29), *is a linear function of* z:

$$\Phi = \Phi_0 + z\Phi_1. \tag{3.46}$$

3.5. A unified formulation of Φ and of the independent variable u

The results of section 3.4 (see (3.41)) suggest introducing two non-symmetric tensors:

$$\hat{G}^{ij} \equiv G^{ij} + \varepsilon^{ij} G_3 \sqrt{\Delta}/2 \qquad \hat{X}^{ij} \equiv X^{ij} + \varepsilon^{ij} k_0 \sqrt{\Delta} \tag{3.47}$$

which satisfy the relations

$$\hat{X}^{ia} \hat{X}_{ja} = -A_6 \delta^i_j \qquad \hat{G}^{ia} \hat{G}_{ja} = 0 \tag{3.48}$$

and hence $\det\left(\hat{G}^{ij}\right) = 0$.

From the linearity of the z-dependence of $\xi'_\pm$, we can write

$$\begin{aligned} \xi'^1 &\propto (z-a); & \xi'^1_- &\propto (z-b) \\ \xi'^2 &\propto (z-a); & \xi'^2_- &\propto (z-\beta) \end{aligned} \tag{3.49}$$

where a, b, α, β are functions of (ξ^1, ξ^2), and then, using (3.37),

$$\begin{aligned} \hat{G}^{11} &= a^{11}(z-a)(z-b) \\ \hat{G}^{12} &= a^{12}(z-a)(z-\beta) \\ \hat{G}^{21} &= a^{21}(z-\alpha)(z-b) \\ \hat{G}^{22} &= a^{22}(z-\alpha)(z-\beta) \end{aligned} \tag{3.50}$$

where a^{ij} are functions of (ξ^1, ξ^2) as well satisfying $\det(a^{ij}) = 0$.

Now, from (3.24), (3.25) and (3.27), $(\xi'^1)^2$ may be written as

$$2A_6(\xi'^1)^2 = P^{11} + \sqrt{\Delta} Q^{11} = 2(\hat{X}^{11}\hat{G}^{12} - \hat{X}^{12}\hat{G}^{11}) \tag{3.51}$$

so the condition that it be proportional to $(z-a)^2$ determines the ratio $\hat{X}^{12}/\hat{X}^{11}$. The ratios of all $\hat{X}$ components can thus be found through the consideration of the corresponding identities relating to $\left(\xi'^i_\pm\right)^2$, and we obtain

$$\begin{aligned} \hat{X}^{11} &= (h/\sqrt{\Delta})a^{11}(a-b) \\ \hat{X}^{12} &= (h/\sqrt{\Delta})a^{12}(a-\beta) \\ \hat{X}^{21} &= (h/\sqrt{\Delta})a^{21}(\alpha-b) \\ \hat{X}^{22} &= (h/\sqrt{\Delta})a^{22}(\alpha-\beta) \end{aligned} \tag{3.52}$$

where h is *a priori* arbitrary and the factor $\sqrt{\Delta}$ has been introduced for convenience.

Thus, $\hat{X}^{ij}$ is proportional to the square root of the discriminant of $\hat{G}^{ij}$.

We also note that, on the locus $\Delta = 0$, where $\xi'_- = \xi'_+$ and $\xi'^2/\xi'^1 = -\partial_1\Delta/\partial_2\Delta$ is independent of z, we must have $a = b = \alpha = \beta$, so that the discriminant of $\hat{G}^{ij}$ has a factor Δ—in addition to the factor $\left(\hat{X}^{ij}\right)^2$. But, $\hat{G}^{11}$ being a polynomial of degree 7, its discriminant is of degree 14, as is the product $\Delta(\hat{X}^{11})^2$—therefore the two quantities are proportional, i.e., the factor h is a constant, which we may take to be unity without loss of generality.

The following relation, which is deduced from (3.48), is also of interest:

$$A_6 = -(h^2/\Delta)a^{11}a^{22}(a-\alpha)(b-\beta). \tag{3.53}$$

If we now calculate the product $\xi'^1\xi'^2$, using an identity of the type of (3.51), we obtain

$$\frac{\xi'^1\xi'^2}{(z-a)(z-\alpha)} = (h/\sqrt{\Delta})a^{11}a^{22}\,(b-\beta)\,/A_6 = \frac{(\sqrt{\Delta}/h)}{(\alpha-a)}. \tag{3.54}$$

That result has a direct implication on the nature of the independent variable u and its relation with the function Φ. Let us consider a differential system of the form

$$\xi'(u) = A(z-a) \qquad \eta'(u) = B(z-\alpha)$$

(where z is a parameter), integrable in the form (3.29), (3.46) and satisfying (3.54):

$$AB = \frac{\sqrt{\Delta}}{(\alpha - a)},$$

we obtain

$$\frac{\mathrm{d}\Phi_0 \wedge \mathrm{d}\Phi_1}{\mathrm{d}\xi \wedge \mathrm{d}\eta} = \frac{AB}{\Delta}(\alpha - a) = \frac{1}{\sqrt{\Delta}}. \tag{3.55}$$

On the other hand, for any differential system integrable in the form (3.29), the independent variable u, viewed as a function of (ξ, η), must have the general form

$$\mathrm{d}u = \frac{1}{2}\left(\frac{\mathrm{d}\xi}{\xi'} + \frac{\mathrm{d}\eta}{\eta'}\right) + \lambda\,\mathrm{d}\Phi \tag{3.56}$$

(where λ is some function of the coordinates), whence

$$\frac{\mathrm{d}\Phi \wedge \mathrm{d}u}{\mathrm{d}\xi \wedge \mathrm{d}\eta} = \frac{1}{\sqrt{\Delta}}. \tag{3.57}$$

Comparing (3.55) and (3.57), we see that u can be identified with Φ_1, i.e. $\partial\Phi/\partial z$ (assuming a finite value for the particular value of z under consideration), to which one may add an arbitrary multiple of Φ.

4. The Painlevé property

A differential system is said to have the Painlevé property if its generic solution is free of movable singularities other than poles. Singularities may only occur at points where the expression of the derivatives becomes non-holomorphic; and in the case of the system represented by the differential equations (3.1) and (3.24), they are the points where ξ' either has a branch-point or becomes infinite. This may occur in the following four cases only:

(a) When $A_6 = 0$.
(b) When $\Delta = 0$.
(c) When $P^{ij} + \sqrt{\Delta}Q^{ij} = 0$.
(d) When one or both of the coordinates (ξ, η) becomes infinite.

Let us examine in turn these four possibilities.

4.1. Case (a): $A_6 = 0$

From (3.35) and (3.37) one easily deduces the identity

$$(P^{ij})^2 - \Delta(Q^{ij})^2 = 4A_6G^{ii}G^{jj} \tag{4.1}$$

while, from the definitions (3.25), (26) of P and Q, one also finds

$$\left(P^{ia}Q_a^j + P^{ja}Q_a^i\right) = 4A_6G_3G^{ij}. \tag{4.2}$$

Assuming that A_6 vanishes, P and Q are thus proportional (from (4.2)), and then, from (4.1)

$$P^{ij} = \varepsilon\sqrt{\Delta}Q^{ij} \qquad (\varepsilon = \pm 1).$$

Let us first consider the case $\varepsilon = +1$.

Then $\xi'^i\xi'^j = \sqrt{\Delta}Q^{ij}/A_6 \to \infty$, and the slope of the integral curve, or trajectory, is $\frac{\mathrm{d}\eta}{\mathrm{d}\xi} = Q^{12}/Q^{11} = Q^{22}/Q^{12}$; on the other hand, we have found (see section 3.3) that the quantity $Q^{ia}\partial_a A_6$ vanishes with A_6—therefore the slope coincides with that of the curve (A_6). Then, according to its definition (3.29), Φ stays constant along (A_6), which is therefore one of the integral curves, and not a part of their envelope (which is the locus $\Delta = 0$). Being a particular solution, it is not relevant to the Painlevé analysis, which deals with the properties of the generic solution.

Turning now to the case $\varepsilon = -1$, we find that the derivatives have the indeterminate form $\xi'^i = 0/0$ at the point (ξ_0, η_0) considered. The slope $\frac{d\eta}{d\xi} = \hat{G}^{21}/\hat{G}^{11}$ differs in general from that of (A_6), which is $\hat{G}^{12}/\hat{G}^{11}$. The quantity $P^{ij} + \sqrt{\Delta}Q^{ij}$, being the only vanishing factor on the left-hand side of equation (4.1), must have a factor (A_6), which cancels out with the denominator of equation (3.24), so that ξ'^i is in fact finite, hence regular, at that point.

4.2. *Case (b):* $\Delta = 0$

Then, as a consequence of (3.1), (3.9) and (3.10), the slope $\frac{d\eta}{d\xi} = -\partial_\xi \Delta/\partial_\eta \Delta$, so that the locus $\Delta = 0$ is the envelope of trajectories; as a consequence, Δ is locally proportional to $(\xi - \xi_0)^2$, and $\sqrt{\Delta}$ in fact has no branch point at that point.

The case of the conic points, which lie at the double points of the locus $\Delta = 0$, must also be considered. Let us choose a coordinate system (x, y) where two branches of the locus, (Π_1) and (Π_2) say, locally coincide with the coordinate axes; the corresponding conic point K_{12} is at the origin. From the fact that the tensor G^{ij} must vanish at conic points, together with its other general properties (section 3.2), it must locally have the form $G_{11} \propto y$; $G_{12} = 0$; $G_{22} \propto x$ and the discriminant $\Delta \propto xy$; so that the differential equation (3.1) becomes $x\,dy^2 - k\,dx^2 = 0$, where k is a constant. The integral curves are the family of parabolae: $(kx - y)^2 - 2a(kx + y) + a^2 = 0$, which are tangential to (Π_1) and (Π_2); thus, the general solution does not pass through the conic points. That result is in agreement with the property that the general solution does not intersect the conic (A_2) either (see section 4.1), which may be viewed as the locus on the (ξ, η) plane of the conic point K_0 at infinity along the vertical axis. (The integral curves that do cross (A_2) belong to a different branch of the general solution—the case $\varepsilon = -1$ in section 4.1—and do so only in projection on the (ξ, η) plane: the corresponding curves on the quartic surface (Σ) do not pass through K_0).

4.3. *Case (c):* $P^{ij} + \sqrt{\Delta}Q^{ij} = 0$

Then either ξ'^1 or ξ'^2 vanishes, and the corresponding $P^{ii} + \sqrt{\Delta}Q^{ii}$ must be zero.

Neglecting the case $A_6 = 0$, already considered in section 4.1, the identity (4.1) entails that if $P^{ii} + \sqrt{\Delta}Q^{ii}$ vanishes then so does G^{ii}, and the former quantity must be proportional to $(G^{ii})^2$. Then, the vanishing ξ'^i is proportional to the polynomial function G^{ii} and thus has no branch point.

4.4. *Case (d):* $\xi^i \to \infty$

That case can be eliminated through a linear coordinate transformation that changes the line at infinity into another straight line. In the reduced form of the system, for example (section 2.2), the line at infinity is one of the diagonals of the quadrangle, and it can be changed into another diagonal without changing the form of the system. In other words, there can be no branch points on the diagonal at infinity, because there are none on the other two diagonals.

5. Conclusion

Starting with a quartic surface (Σ) of the class defined in section 2, we have shown how to associate with it a one-parameter family of symmetric tensors G, polynomial functions of the coordinates, which may be viewed as playing the role of a metric on (Σ).

For each value of the parameter, the null curves of the corresponding metric are the solutions of a differential system, which we have shown to have the Painlevé property and to be integrable by quadratures as well.

Differential systems of exactly the same form have been shown (Gaffet 2006) to represent the evolution of a certain class of rotating and precessing gas clouds which obey the system of Euler equations of gasdynamics (Ovsiannikov 1956, Dyson 1968).

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